Developments in random matrix theory

2003 J. Phys. A: Math. Gen. 36 R1

(http://iopscience.iop.org/0305-4470/36/12/201)

J. Phys. A: Math. Gen. **36** (2003) R1–R10 PII: S0305-4470(03)60304-5

INTRODUCTORY REVIEW

Developments in random matrix theory

P J Forrester[1], **N C Snaith**[2] **and J J M Verbaarschot**[3]

[1] Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia
[2] School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, UK
[3] Department of Physics and Astronomy, SUNY Stony Brook, Stony Brook, NY 11790, USA

E-mail: P.Forrester@ms.unimelb.edu.au, N.C.Snaith@bristol.ac.uk and jacobus.verbaarschot@stonybrook.edu

Received 4 March 2003, in final form 10 March 2003
Published 20 March 2003
Online at stacks.iop.org/JPhysA/36/R1

Abstract

In this introduction to the *Journal of Physics A* special issue on random matrix theory, we give a review of the main historical developments in random matrix theory. A short summary of the papers that appear in this special issue is also given.

1. Introduction

Random matrix theory has matured into a field with applications in many branches of physics and mathematics. A large number of physicists and mathematicians have been fascinated by this subject that has turned out to be surprisingly rich and far reaching. Paraphrasing Dyson, random matrix theory is a new kind of statistical mechanics where the realization of the system is not relevant. Instead of having an ensemble of states we have an ensemble of Hamiltonians. Ergodicity is now the equivalence of spectral averaging and the averaging over this ensemble.

Random matrix theory has been particularly successful in three areas: first, in describing level correlations on the scale of the average level spacing; second, in providing the generating function for combinatorial factors of planar diagrams and, third, as an exactly solvable model with intricate connections to the theory of integrable systems. One of the reasons for the success of random matrix theory is universality: eigenvalue correlations on the scale of the average level spacing do not depend on the probability distribution. This property is at the very foundation of random matrix theory. It suggests that random matrix theory correlations of eigenvalues should be the rule rather than the exception. However, the most important reason for studying random matrix theory is that its predictions do occur in nature in systems as varied as nuclear energy levels, zeros of the Riemann ζ function and sound waves in quartz crystals. Another of the roles of random matrix theory is that the large-N limit of its partition function is a generating function for planar diagrams which have played an important role in quantum field theory. For example, they are the leading contributions to QCD with a large number of colours, and they are dual to triangulations of a random surface and thus describe two-dimensional quantum gravity. In addition to this, random matrix theory has attracted a great deal of interest because of the mathematical challenges it poses. The problems are highly nontrivial, but, with sufficient effort, many of the questions that arise in this field can be answered in full.

It is hard not to be fascinated by random matrix theory. Everyone who works in this field has experienced the amazement of obtaining truly universal behaviour by diagonalizing a large random matrix. Nowadays, as we can see from the contributions to this special issue, the excitement about the subject is still as much alive as when it was first created. In this introduction to the special issue we give a short summary of the history of random matrix theory. The historical perspective is certainly coloured by our personal experience, and, for a somewhat different perspective, we refer the reader to the introduction in the review by the Heidelberg group [1].

2. History of random matrix theory

Random matrix theories have fascinated both mathematicians and physicists since they were first introduced in mathematical statistics by Wishart in 1928 [2]. After a slow start, the subject gained prominence when Wigner [3] introduced the concept of statistical distribution of nuclear energy levels in 1950. However, it took until 1955 before Wigner [4] introduced ensembles of random matrices. In that paper he also introduced the large-N expansion and realized that the leading order contribution to the expectation values of moments of the random Hamiltonian is given by planar diagrams. In 1956, Wigner [5] derived the Wigner surmise from the level spacing distribution of an ensemble of 2×2 matrices after level repulsion had been predicted by Landau and Smorodinksy [6] and observed by Gurevich and Pevsner [7]. The idea of invariant random matrix ensembles was introduced in physics by Porter and Rosenzweig [8] after it had appeared earlier in the mathematical literature. A mathematically rigorous analysis of spacing distributions was first given by Gaudin [9] and Mehta [10]. To analyse the eigenvalue density Mehta [11] invented the orthogonal polynomial method.

The mathematical foundations of random matrix theory were established in a series of beautiful papers by Dyson [12–16]. He introduced the classification of random matrix ensembles according to their invariance properties under time reversal [12, 16]. As we all know, only three different possibilities exist: a system is not time reversal invariant, or a system is time reversal invariant with the square of the time reversal invariance operator either equal to 1 or -1. The matrix elements of the corresponding random matrix ensembles are complex, real and self-dual quaternion, respectively, which from a mathematical viewpoint exhaust the distinct real commutative normed division algebras, or in effect number systems. The corresponding invariant Gaussian ensembles of Hermitian random matrices are known as the Gaussian unitary ensemble (GUE), the Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE), in that order.

Dyson [12] also formulated the underlying philosophy of random matrix theory. In his words, ‘What is here required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself. We picture a complex nucleus as a “black box” in which a large number of particles are interacting according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally possible’. This was made more precise by Balian [17], who obtained the Gaussian random matrix ensembles from minimizing the information entropy.

A second important result of Dyson’s papers [14, 15] was the relation between random matrix theory and the theory of exactly integrable systems: the partition function of a random matrix ensemble is equivalent to the partition function of a log-potential Coulomb gas in one dimension at three special temperatures, each with solvability properties not shared for general temperatures. Moreover the evolution of the eigenvalues of parameter-dependent extensions of the Gaussian ensembles was shown to be controlled by a Fokker–Planck operator which also specifies the Brownian evolution of the Coulomb gas. These results were further explored by Sutherland [18] when he realized that the Calogero–Sutherland quantum many body system [19, 18], for which the Hamiltonian can be constructed from N independent commuting operators and so is integrable, is mathematically equivalent to Dyson's Brownian motion model. The relation between random matrix theory and integrable systems is discussed extensively in the monograph by Forrester [20]. A review of one-dimensional integrable systems that touches on many ideas that also appear in random matrix theory is given in the book by Korepin *et al* [21]. A third idea that appeared in Dyson's paper [12] is the application of Shannon's information entropy to random matrix spectra.

The early developments in random matrix theory are well summarized in the first edition of the monograph by Mehta [22]. This has been a very influential book containing many mathematical details which have been proved to be extremely useful over the years. A second significant book is by Porter [23]. It contains reprints of the important papers on random matrix theory that were written before 1965.

About the same time as the early development of random matrix theory in nuclear physics, the field of disordered systems was born from the work by Anderson [24] on the localization of wavefunctions in one-dimensional disordered systems. He considered a one-dimensional lattice with a random potential at each lattice point. He found that the eigenfunctions of this system are exponentially localized. His work had a strong impact on both experimental and theoretical solid state physics. Another early application of random matrix theory is the theory of small metallic particles by Gorkov and Eliasberg [25], which nowadays would be part of mesoscopic physics.

Random matrix theory, which was first formulated in mathematical statistics, continued to develop in mathematics independently of the developments in physics. Important results with regard to the integration measure of invariant random matrix ensembles were obtained by Hua [26]. His results of more than a decade of work are summarized in his book that appeared in 1959 but which remained largely unknown. Only a small number of mathematicians worked on integrals that appear in random matrix theory. One very important result was obtained by Harish-Chandra [27], who evaluated a unitary matrix integral that is now known as the Harish-Chandra–Itzykson–Zuber integral [27, 28]. Zinn-Justin and Zuber [29] review this topic in the present special issue. Also the work of Selberg [30] is well known, not in the least because Madan Lal Mehta devoted a chapter of the second edition of his book [31] to this subject. Another noteworthy contribution is the introduction of zonal polynomials by James [32]. The 1982 book of Muirhead [33] ties together matrix integrals and zonal polynomials as they are relevant in mathematical statistics. Girko has written a number of mathematical books (see, e.g., [34]) relating to analytic properties of the eigenvalue distribution of large random matrices. Voiculescu [35] used random matrices as a primary example of the concept of free non-commutative random variables in operator algebras. However, the mathematical literature remained largely unnoticed by physicists until recently.

What is more surprising is that the theory of disordered systems and the application of random matrix theory in nuclear physics proceeded more or less independently until the seminal work by Efetov on the supersymmetric method [36] and its application [37] to the theory of small metallic particles and to localization theory [38] . This is even more remarkable since both the papers by Anderson and Dyson were written at Princeton.

The main developments in random matrix theory in the decade after the appearance of the first edition of Mehta's book were applications to nuclear physics. In particular, the statistical theory of S-matrix fluctuations received a great deal of attention. The first work in this direction dates back to Wigner [39], who simultaneously studied the distribution of the widths and the spacings of nuclear resonances, and to Porter and Thomas [40], who introduced the Porter–Thomas distribution for nuclear decay widths. Correlations of cross-sections at two different energies were considered in [41] and are now known as Ericson fluctuations. The formulation of the theory of S-matrix fluctuations was completed in the work of Agassi *et al* [42] . In this paper the authors introduced resummation techniques which later received much more attention in the field of impurity scattering, introduced earlier in the book by Abrikosov *et al* [43]. The problem of the distribution of poles of S-matrices was also the motivation of Ginibre [44] for introducing what is now known as the Ginibre ensemble with eigenvalues uniformly distributed inside a disc in the complex plane. His paper initiated the subfield of non-Hermitian random matrix theory which is reviewed by Fyodorov and Sommers [45] in this issue.

In 1973 Montgomery [46] made a conjecture for the asymptotic limit of the two-point correlation function of the zeros of the Riemann ζ function on the critical line. Together with Dyson he realized that his conjectured result is the two-point function of the GUE. The connection was extended to higher correlation functions of the Riemann zeros by Hejhal [47] and Rudnick and Sarnak [48], although the full correspondence of the correlation functions with random matrix theory has still not been proved. A heuristic derivation of these results using the Hardy–Littlewood conjecture for the correlation between primes, was given by Bogomolny and Keating in 1995 [49, 50]. Mathematically rigorous results relating the two-point functions for the zeros of families of finite field zeta functions and eigenvalues of random matrices from the classical groups are the topic of the monograph by Katz and Sarnak [51].

The conjectured correspondence of the statistics of these zeros of the Riemann ζ function with the n-point correlation function of random matrix eigenvalues has recently meant that random matrix theory has become very useful for conjecturing quantities in number theory that were previously unattainable by any method. These include mean values of the Riemann zeta function and other L-functions [52–56], the order of vanishing at special values of L-functions [57], as well as discrete moments of the derivative of the Riemann zeta function [58, 59] and the horizontal distribution of the zeros of the derivative [60]. For more details there is a review in this issue by Keating and Snaith [61].

In the period 1975–1985, random matrix theory developed rapidly and became unified with the theory of disordered systems. The first step in this direction was made by Edwards and Anderson [62] who, in their influential paper on spin glasses, introduced the replica trick. This provided a natural framework for a field theoretical formulation of the Anderson model which was introduced a few years later by Wegner in 1979 [63]. In this formulation, symmetries and the spontaneous breaking of symmetries led to a new paradigm in the theory of Anderson localization [64–66]. It was soon realized that the replica formulation only works well for perturbative calculations. This problem was solved by the introduction of the supersymmetric method [36]. In this method the determinants in the generating function of the resolvent are quenched by taking a ratio of two determinants instead of the $n \to 0$ limit of the nth power of the determinant. Relying on an earlier work by Wegner [63] using the replica trick, Efetov showed that the partition function of a disordered system is given by a supersymmetric nonlinear σ-model. He identified a domain of energy differences where the kinetic term of the nonlinear σ model can be neglected. In this domain the two-point correlation functions coincide with the results derived by Dyson. The energy scale below which the partition function is dominated by zero momentum modes is known as the Thouless energy [67].

The supersymmetric method has been very fruitful. Efetov [36] obtained new results for one-dimensional disordered wires. Exact results were obtained for the theory of S-matrix fluctuations [68]. Relations between the orthogonal and symplectic symmetry classes were derived [69] from the supersymmetric partition function. In the subsequent years many more new results were derived by means of the supersymmetric method. Among others, we mention results for parametric correlations [70] where eigenvalue correlations for different values of an external parameter are considered. An elaborate discussion of applications of the supersymmetric method to disordered systems is given in the book by Efetov [71].

Exact results for S-matrix fluctuations were obtained in a completely independent way by a Mexican group [72]. The exact distribution function of S-matrices was found starting from the three assumptions of analyticity, ergodicity and maximizing the information entropy [17]. Another effort in nuclear physics was the introduction of random matrix ensembles that reflected the few-body nature of the interaction. In particular, French and co-workers have pursued this direction of research (see [73] for a review). In this issue Benet and Weidenmueller [74] review recent progress in this field.

A major development was the experimental discovery of universal conductance fluctuations by Webb and Washburn in 1986 [75] after having been predicted theoretically by Altshuler [76] and Stone and Lee [77, 78]. This discovery started the new field of chaotic quantum dots. The transport properties of these quantum dots could be described by the supersymmetric nonlinear σ-model that had been used for the theory of S-matrix fluctuations in compound nuclei. In fact, a compound nucleus is a chaotic quantum dot (see [79, 80, 81] in this issue for reviews).

A few years before the discovery of universal conductance fluctuations, random matrix theory was applied to quantum field theory. Through the work of 't Hooft [82] we know that in the limit of a large number of colours, the QCD partition function is dominated by planar diagrams. This is also the case for the large-N limit of random matrix theory. In [83] this was exploited to calculate the combinatorial factors that enter in the large-N_c limit of QCD by means of random matrix theory. A second innovative idea which appeared in that paper is the formulation of the calculation of the resolvent in random matrix theories as a Riemann–Hilbert problem. This approach has received more attention in the recent mathematical literature [84].

Random matrix theory has had impact on several areas of quantum field theory: lattice QCD, two-dimensional gravity, the Euclidean Dirac spectrum and the Seiberg–Witten [85] solution of two-dimensional supersymmetric gauge theories. An important result is the Eguchi–Kawai [86] reduction. These authors showed that in the limit of a large number of colours, certain gluonic correlation functions of pure Yang–Mills theory can be reduced to an integral over four unitary matrices. In two spatial dimensions this reduction results in an integral over a single unitary matrix which can be evaluated in the large-N limit.

A unitary matrix integral also occurs in the low-energy limit of QCD. Because of the spontaneous breaking of chiral symmetry, its low-energy degrees of freedom are the Goldstone modes which are parametrized by a unitary matrix valued field [87]. Below the Thouless energy for this system the kinetic term of the effective Lagrangian can be neglected and the low-energy limit of the QCD partition function is given by the unitary matrix integral [88]. In this domain the eigenvalues of the Dirac operator are correlated according to a random matrix theory with the additional involutive (chiral) symmetry of the QCD Dirac operator [89, 90]. The same symmetry is also found in two-sublattice disordered systems where hopping only occurs in between the sublattices [91]. The eigenvalue spectrum around zero of these chiral ensembles was first derived in [92]. An important difference between two-sublattice systems and QCD is the topology of the random matrix (i.e. the number of exact zeros) and the fermion determinant. In two-sublattice systems one is only interested in quenched results at zero topology whereas in QCD the fermion determinant and its zero modes are essential. Also, in the case of the chiral ensembles we have three different symmetry classes depending on the reality content of the matrix elements. Most of the work on chiral random matrix theory and its applications to the Dirac spectrum in QCD was done in the second half of the nineties (see [93] for a review).

In the theory of disordered superconductors four more random matrix ensembles can be introduced [94, 95], thus distinguishing a total of ten random matrix ensembles. It was noted by Dyson [96] that each of the three Wigner–Dyson random matrix theories corresponds to a symmetric space. Zirnbauer [97] showed that this observation can be generalized to all ten symmetry classes of random matrix ensembles with a one-to-one correspondence to each of the large families in the Cartan classification of symmetric spaces.

There have been other attempts to derive QCD from a matrix model. Perhaps best known is the induced QCD partition function of Kazakov and Migdal [98] where the lattice gauge field is coupled to an adjoint scalar field. The gauge field can be integrated out by means of the Harish-Chandra–Itzykson–Zuber integral resulting in a partition function for the eigenvalues of the adjoint scalar field. This partition function can be evaluated by saddle point methods in the large-N limit. More recently, it has been shown that that the so-called prepotential of $N = 2$ supersymmetric theory can be derived from the large-N limit of a random matrix theory [99].

The partition function of 2D gravity is a sum over random surfaces which can be described by means of a triangulation [100, 101]. The sum over triangulated surfaces can be written in terms of a random matrix theory partition function. It has been conjectured [102–104] that the double scaling limit of this theory describes the continuum limit of the 2D gravity partition function. This field brought two new ideas into random matrix theory: universality [105–107], i.e. that observables are independent of the probability potential, and the connection with integrable systems. In the context of quantum gravity it is natural to consider an arbitrary polynomial probability potential. Integrable hierarchies were obtained from differential equations in the coefficients of the probability potential [108, 109]. A good review of this topic was given by Di Francesco *et al* [110].

Earlier integrable hierarchies entered in a completely different way. In 1980 it was found by the Kyoto school [111] that the probability of a gap-free interval in the infinite GUE is a τ-function for a completely integrable system specifying the isomonodromy deformation of a coupled system of linear differential equations. This had the consequence that the spacing distribution could be expressed in terms of a Painlevé V transcendent. Later it was found that the distribution of the largest eigenvalue in the GUE is given by the solution of a Painlevé II equation [112]. This development found application in the solution of a long standing mathematical problem: specifying the limiting distribution of the longest increasing subsequence length of a random permutation [113]. In fact the sought distribution is the same as that for the largest eigenvalue in the GUE. The increasing subsequence problem can equivalently be formulated as the polynuclear growth model in 1 + 1 dimension [114], and similar relationships with random matrix fluctuations are also known for certain tiling problems [115].

The question of why random matrix theory works has been addressed from many different points of view. It was realized early on that the detailed properties of eigenvalue correlations do not depend on the specifics of the probability distribution. One important reason for random matrix theory to work is already mentioned in a work of Dyson [12], asserting that if a system is sufficiently complex, the state of the system is no longer important. However, it took until the early eighties before it was realized that the key reason is that the corresponding classical system is chaotic. Although there have been a few earlier studies relating random matrix theory correlations to classical chaos [116, 117], it was formulated explicitly in a ground breaking paper by Bohigas *et al* [118] who, based on a numerical study of the Sinai billiard [119], conjectured that level correlations on the scale of the average level spacing are given by random matrix theory if the corresponding classical system is chaotic. This conjecture has been confirmed for numerous systems. The reverse was also shown numerically to be true: if the system is not completely chaotic, the spectral correlations are not given by the Wigner–Dyson ensembles [120, 121]. Although a complete proof of this conjecture is still lacking, a considerable amount of analytical understanding has been obtained on the basis of a semiclassical analysis [122, 123]. These inter-relations mean that random matrix theory plays an essential role in the study of quantum chaos, a fact which is given prominence in the books by Haake [124] and Stoeckmann [125].

In this short historical overview we have seen that random matrix theory has been applied to wide ranging fields. Its scope has by far not yet been exhausted as illustrated by recent publications that are as varied as applications to financial correlations [126, 127] and wireless communication [128].

Acknowledgments

This work was partially supported by the Australian Research Council (PJF), the Royal Society Dorothy Hodgkin Research Fellowship scheme (NS) and the US DOE grant DE-FG-88ER40388 (JJMV).

References

[1] Guhr T, Mueller-Groeling A and Weidenmueller H A 1998 Random-matrix theories in quantum physics: common concepts *Phys. Rep.* **299** 189
[2] Wishart J 1928 Generalized product moment distribution in samples *Biometrika* A **20** 32
[3] Wigner E P 1951 On the statistical distribution of the widths and spacings of nuclear resonance levels *Proc. Camb. Phil. Soc.* **47** 790
[4] Wigner E P 1955 Characteristic vectors of bordered matrices with infinite dimensions *Ann. Math.* **62** 548
[5] Wigner E P 1957 Statistical properties of real symmetric matrices with many dimensions *Can. Math. Congr. Proc.* (Toronto: University of Toronto Press) p 174
[6] Landau L and Smorodinsky Ya 1955 *Lectures on the Theory of the Atomic Nucleus* (Moscow: State Technical-Theoretical Literature Press) p 92
[7] Gurevich I I and Pevsner M I 1957 Repulsion of nuclear levels *Nucl. Phys.* **2** 575
[8] Porter C E and Rosenzweig N 1960 Statistical properties of atomic and nuclear spectra *Suomalaisen Tiedeakatemian Toimituksia AVI* 44
[9] Gaudin M 1961 Sur la loi limite de l'espacement des valeurs propres d'une matrice aléatoire *Nucl. Phys.* **25** 447
[10] Mehta M L 1960 On the statistical properties of level spacings in nuclear spectra *Nucl. Phys.* **18** 395
[11] Mehta M L 1960 L'emploi del polynomes orthogonaux pour calculer certain determinants *Rapport S PH (Saclay)* 658
[12] Dyson F J 1962 Statistical theory of the energy levels of complex systems, I *J. Math. Phys.* **3** 140
[13] Dyson F J 1962 Statistical theory of the energy levels of complex systems, II *J. Math. Phys.* **3** 157
[14] Dyson F J 1962 Statistical theory of the energy levels of complex systems, III *J. Math. Phys.* **3** 166
[15] Dyson F J 1962 A Brownian motion model for the eigenvalues of a random matrix *J. Math. Phys.* **3** 1191
[16] Dyson F J 1962 The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics *J. Math. Phys.* **3** 1200
[17] Balian R 1968 Random matrices and information theory *Nuovo Cimento* B **57** 183
[18] Sutherland B 1971 Exact results for a quantum many-body system in one dimension *Phys. Rev.* A **4** 2019
[19] Calogero F 1971 Solution of one-dimensional *N*-body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419
[20] Forrester P J Log-gases and random matrices webpage http://www.ms.unimelb.edu.au/~matjpf/matpjf.html

[21] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[22] Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic)
[23] Porter C E 1965 *Statistical Theory of Spectra: Fluctuations* (New York: Academic)
[24] Anderson P W 1958 Absence of diffusion in certain random lattices *Phys. Rev.* **109** 1492
[25] Gorkov L P and Eliasberg G M 1965 Minute metallic particles in an electromagnetic field *Sov. Phys.–JETP* **21** 940
[26] Hua L K 1958 *Harmonic Analysis of Functions of Many Complex Variables in Classical Domains* (Peking: Science Press)
[27] Harish-Chandra 1957 Differential operators on a semi-simple Lie algebra *Am. J. Math.* **79** 85
[28] Itzykson C and Zuber J-B 1980 Planar approximation 2 *J. Math. Phys.* **21** 411
[29] Zinn-Justin P and Zuber J-B 2003 On some integrals over the $U(N)$ unitary group and their large N limit *J. Phys. A: Math. Gen.* **36** 3173–93 (*Preprint* math-ph/0209019)
[30] Selberg A 1944 Bemerkninger om et multipelt integral *Nord. Mat. Tid.* **26** 71
[31] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[32] James A T 1961 Zonal polynomials of real positive definite symmetric matrices *Ann. Math.* **74** 456
[33] Muirhead R J 1982 *Aspects of Multivariate Statistical Theory* (New York: Wiley)
[34] Girko V L 1990 *Theory of Random Determinants* (Dordrecht: Kluwer)
[35] Voiculescu D 1991 Limit laws for random matrices and free products *Invent. Math.* **104** 201
[36] Efetov K B 1983 Supersymmetry and the theory of disordered metals *Adv. Phys.* **32** 53
[37] Efetov K B 1982 Level spacing distribution in small metallic particles *J. Phys. G: Nucl. Phys.* **15** L909
[38] Efetov K B 1982 Supersymmetric method in the localization theory *Zh. Eksp. Teor. Fiz.* **82** 872
[39] Wigner E P 1951 On a class of analytic functions from the quantum theory of collisions *Ann. Math.* **53** 36
[40] Porter C E and Thomas R G 1956 Fluctuations of nuclear reaction widths *Phys. Rev.* **104** 483
[41] Ericson T 1960 Fluctuations of nuclear cross-sections in the continuum region *Phys. Rev. Lett.* **5** 430
[42] Agassi D, Mantzouranis G and Weidenmueller H A 1975 The statistical theory of nuclear reactions for strongly overlapping resonances as a theory of transport phenomena *Phys. Rep.* C **22** 145
[43] Abrikosov A A, Gorkov L P and Dzyaloshinski I E 1963 *Methods of Quantum Field Theory in Statistical Physics* (Engelwood Cliffs, NJ: Prentice-Hall)
[44] Ginibre J 1965 Statistical ensembles of complex, quaternion and real matrices *J Math. Phys.* **6** 440
[45] Fyodorov Y V and Sommers H-J 2003 Random Matrices close to Hermitian or unitary: overview of methods and results *J. Phys. A: Math. Gen.* **36** 3303–47 (*Preprint* nlin.CD/0207051)
[46] Montgomery H L 1973 The pair correlation of the zeta function *Proc. Symp. Pure Math.* **24** 181
[47] Hejhal D A 1994 On the triple correlation of zeros of the zeta function *Int. Math. Res. Not.* **7** 293
[48] Rudnick Z and Sarnak P 1996 Zeros of principle L-functions and random matrix theory *Duke Math. J.* **81** 269
[49] Bogomolny E B and Keating J P 1995 Random matrix theory and the Riemann zeros I: three- and four-point correlations *Nonlinearity* **8** 1115
[50] Bogomolny E B and Keating J P 1996 Random matrix theory and the Riemann zeros II: n-point correlations *Nonlinearity* **9** 911
[51] Katz N M and Sarnak P 1999 *Random Matrices, Frobenius Eigenvalues and Monodromy* (Providence, RI: American Mathematical Society)
[52] Conrey J B, Farmer D W, Keating J P, Rubinstein M O and Snaith N C 2002 Integral moments of zeta- and L-functions *Preprint* math.nt/0206018
[53] Conrey J B, Farmer D W, Keating J P, Rubinstein M O and Snaith N C 2002 Autocorrelation of random matrix polynomials *Preprint* math-ph/0208007 (*Commun. Math. Phys.* at press)
[54] Keating J P, Linden N and Rudnik Z 2003 Random matrix theory, the exceptional Lie groups and L-functions *J. Phys. A: Math. Gen.* **36** 2933–44 (*Preprint* math-ph/0210058)
[55] Keating J P and Snaith N C 2000 Random matrix theory and $\zeta(1/2 + it)$ *Commun. Math. Phys.* **214** 57
[56] Keating J P and Snaith N C 2000 Random matrix theory and L-functions at $s = 1/2$ *Commun. Math. Phys.* **214** 91
[57] Conrey J B, Keating J P, Rubinstein M O and Snaith N C 2001 On the frequency of vanishing of quadratic twists of modular L-functions *Number Theory for the Millennium: Proc. of the Millennial Conf. on Number Theory* ed B C Berndt (Boston: A K Peters) (*Preprint* math.nt/0012043)
[58] Hughes C P 2003 Random matrix theory and discrete moments of the Riemann zeta function *J. Phys. A: Math. Gen.* **36** 2907–17 (*Preprint* math.nt/0207236)
[59] Hughes C P, Keating J P and O'Connell N 2000 Random matrix theory and the derivative of the Riemann zeta function *Proc. R. Soc.* A **456** 2611

[60] Mezzadri F 2003 Random matrix theory and the zeros of $\zeta'(s)$ *J. Phys. A: Math. Gen.* **36** 2945–62 (*Preprint* math-ph/0207044)
[61] Keating J P and Snaith N 2003 Random matrices and *L*-functions *J. Phys. A: Math. Gen.* **36** 2859–81
[62] Edwards S F and Anderson P W 1975 Theory of spin glasses *J. Phys. F: Met. Phys.* **5** 965
[63] Wegner F J 1979 Mobility edge problem—continuous symmetry and a conjecture *Z. Phys.* B **35** 209
[64] Efetov K B, Larkin A I and Khemlnitsky D E 1980 Interaction of diffusion modes in the theory of localization *Zh. Eksp. Teor. Fiz.* **79** 1120
[65] McKane A J and Stone M 1981 Localization as an alternative to Goldstone theorem *Ann. Phys., NY* **131** 36
[66] Schaefer L and Wegner F J 1980 Disordered system with *N*-orbitals per site-Lagrange formulation, hyperbolic symmetry, and Goldstone modes *Z. Phys.* B **38** 113
[67] Thouless D J 1977 Maximum metallic resistance in thin wires *Phys. Rev. Lett.* **39** 1167
[68] Verbaarschot J J M, Weidenmueller H A and Zirnbauer M R 1985 Grassmann integration in stochastic quantum physics: the case of compound-nucleus scattering *Phys. Rep.* **129** 367
[69] Wegner F J 1983 Algebraic derivation of symmetry-relations for disordered electronic systems *Z. Phys.* B **49** 297
[70] Simons B D and Altshuler B L 1993 Universal velocity correlations in disordered and chaotic systems *Phys. Rev. Lett.* **70** 4063
[71] Efetov K B 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[72] Mello P A, Pereyra P and Seligman T 1985 Information-theory and statistical nuclear reactions 1. General theory and applications to few-channel problems *Ann. Phys., NY* **161** 254
[73] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 Random-matrix physics—spectrum and strength fluctuations *Rev. Mod. Phys.* **53** 385
[74] Benet L and Weidenmüller H A 2003 Review of the k-Body embedded ensembles of Gaussian random matrices *J. Phys. A: Math. Gen.* **36** 3569–93 (*Preprint* cond-mat/0207656)
[75] Webb R A, Washburn S, Umbach C P and Laibwitz R B 1985 Observation of H/E Ahoronov–Bohm oscillations in normal-metal rings *Phys. Rev. Lett.* **54** 2696
[76] Altshuler B L 1985 Fluctuations in the extrinsic conductivity of disordered conductors *Pis'ma Zh. Eksp. Teor. Fiz.* **41** 530
[77] Stone A D 1985 Magnetoresistance fluctuations in mesoscopic wires and rings *Phys. Rev. Lett.* **54** 2692
[78] Lee P A and Stone A D 1985 Universal conductance fluctuations in metals *Phys. Rev. Lett.* **55** 1622
[79] Beenakker C W J 1997 Random-matrix theory of quantum transport *Rev. Mod. Phys.* **69** 731
[80] Alhassid Y 2000 The statistical theory of quantum dots *Rev. Mod. Phys.* **72** 895
[81] Polianski M L and Brouwer P W 2003 Scattering matrix ensemble for time-dependent transport through a chaotic quantum dot *J. Phys. A: Math. Gen.* **36** 3215–36 (*Preprint* cond-mat/0208408)
[82] 't Hooft G 1974 A two-dimensional model for mesons *Nucl. Phys.* B **75** 461
[83] Brezin E, Itzykson C, Parisi G and Zuber J B 1978 Planar diagrams *Commun. Math. Phys.* **59** 35
[84] Deift P P, Kriecherbauer T, McLaughlin K T R, Venakides S and Zhou X 1999 Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory *Commun. Pure Appl. Math.* **52** 1335
[85] Seiberg N and Witten E 1994 Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory *Nucl. Phys.* B **426** 19
Seiberg N and Witten E 1994 *Nucl. Phys.* B **430** 485 (erratum) (*Preprint* hep-th/9407087)
[86] Eguchi T and Kawai H 1982 Reduction of dynamical degrees of freedom in the large *N* gauge theory *Phys. Rev. Lett.* **48** 1063
[87] Weinberg S 1966 Pion scattering lengths *Phys. Rev. Lett.* **17** 616
[88] Gasser J and Leutwyler H 1987 Light quarks at low temperatures *Phys. Lett.* B **184** 83
[89] Shuryak E V and Verbaarschot J J 1993 Random matrix theory and spectral sum rules for the Dirac operator in QCD *Nucl. Phys.* A **560** 306 (*Preprint* hep-th/9212088)
[90] Verbaarschot J J 1994 The spectrum of the QCD Dirac operator and chiral random matrix theory: the threefold way *Phys. Rev. Lett.* **72** 2531 (*Preprint* hep-th/9401059)
[91] Gade R 1993 Anderson localization for sublattice models *Nucl. Phys.* B **398** 398
[92] Forrester P J 1993 The spectrum edge of random matrix ensembles *Nucl. Phys.* B **402** 709
[93] Verbaarschot J J and Wettig T 2000 Random matrix theory and chiral symmetry in QCD *Annu. Rev. Nucl. Part. Sci.* **50** 343 (*Preprint* hep-ph/0003017)
[94] Oppermann R 1990 Anderson localization problems in gapless superconducting phases *Physica* A **167** 301
[95] Altland A and Zirnbauer M R 1996 Random matrix theory of a chaotic Andreev quantum dot *Phys. Rev. Lett.* **76** 3420
[96] Dyson F J 1970 Correlations between the eigenvalues of a random matrix *Commun. Math. Phys.* **19** 235

[97] Zirnbauer M R 1996 Riemannian symmetric superspaces and their origin in random-matrix theory *J. Math. Phys.* **37** 4986
[98] Kazakov V A and Migdal A A 1993 Induced QCD at large *N Nucl. Phys.* B **397** 214 (*Preprint* hep-th/9206015)
[99] Dijkgraaf R and Vafa C 2002 Matrix models, topological strings, and supersymmetric gauge theories *Nucl. Phys.* B **644** 3 (*Preprint* hep-th/0206255)
[100] David F 1985 Planar diagrams, two-dimensional lattice gravity and surface models *Nucl. Phys.* B **257** 45
[101] Kazakov V A, Migdal A A and Kostov I K 1985 Critical properties of randomly triangulated planar random surfaces *Phys. Lett.* B **157** 295
[102] Douglas M R and Shenker S H 1990 Strings in less than one-dimension *Nucl. Phys.* B **335** 635
[103] Brezin E and Kazakov V A 1990 Exactly solvable field theories of closed strings *Phys. Lett.* B **236** 144
[104] Gross D J and Migdal A A 1990 Nonperturbative two-dimensional quantum gravity *Phys. Rev. Lett.* **64** 127
[105] Ambjorn J, Jurkiewicz J and Makeenko Y M 1990 Multiloop correlators for two-dimensional quantum gravity *Phys. Lett.* B **251** 517
[106] Bowick M J and Brezin E 1991 Universal scaling of the tail of the density of eigenvalues in random matrix models *Phys. Lett.* B **268** 21
[107] Akemann G, Damgaard P H, Magnea U and Nishigaki S 1997 Universality of random matrices in the microscopic limit and the Dirac operator spectrum *Nucl. Phys.* B **487** 721 (*Preprint* hep-th/9609174)
[108] Douglas M R 1990 Strings in less than one-dimension and the generalized K-D-V hierarchies *Phys. Lett.* B **238** 176
[109] Dijkgraaf R, Verlinde H and Verlinde E 1991 Loop equations and Virasoro constraints in nonperturbative 2-d quantum gravity *Nucl. Phys.* B **348** 435
[110] Di Francesco P, Ginsparg P and Zinn-Justin J 1995 2-D Gravity and random matrices *Phys. Rep.* **254** 1 (*Preprint* hep-th/9306153)
[111] Jimbo M, Miwa T, Môri Y and Sato M 1980 Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent *Physica* D **1** 80
[112] Tracy C A and Widom H 1994 Level spacing distributions and the Airy kernel *Commun. Math. Phys.* **159** 151
[113] Baik J, Deift P and Johannsson K 1999 On the distribution of the length of the longest increasing subsequence of random permutations *J. Am. Math. Soc.* **36** 1119
[114] Prähofer M and Spohn H 2000 Universal distributions for growth processes in one dimension and random matrices *Phys. Rev. Lett.* **84** 4882
[115] Johansson K 2002 Non-intersecting paths, random tilings and random matrices *Prob. Theory Rel. Fields* **123** 225
[116] Zaslavsky G M 1981 Stochasticity in quantum systems *Phys. Rep.* **80** 157
[117] Berry M V 1981 Quantizing a classically ergodic system: Sinai's billiard and the KKR method *Ann. Phys., NY* **131** 163
[118] Bohigas O, Giannoni M J and Schmit C 1984 Characterization of chaotic quantum spectra and universality of level fluctuation laws *Phys. Rev. Lett.* **52** 1
[119] Sinai Ya G 1970 Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards *Russ. Math. Surv.* **25** 137
[120] Haller E, Koppel H and Cederbaum L S 1984 Uncovering the transition from regularity to irregularity in a quantum system *Phys. Rev. Lett.* **52** 1665
[121] Seligman T, Verbaarschot J J M and Zirnbauer M R 1984 Quantum chaos and the transition from chaotic to regular motion *Phys. Rev. Lett.* **53** 215
[122] Berry M V 1985 Semiclassical theory of spectral rigidity *Proc. R. Soc.* A **400** 229
[123] Bogomolny E B and Keating J P 1996 Gutzwiller's trace formula and spectral statistics *Phys. Rev. Lett.* **77** 1472
[124] Haake F 1999 *Quantum Signatures of Chaos* 2nd edn (Berlin: Springer)
[125] Stoeckmann H-J 1999 *Quantum Chaos: An Introduction* (Cambridge: Cambridge University Press)
[126] Laloux L, Cizeau P, Bouchard J-P and Potters M 1999 Noise dressing of financial correlation matrices *Phys. Rev. Lett.* **83** 1467
[127] Plerou V, Gopikrishnan P, Rosnow B, Nunes Amaral L A and Stanley H E 1999 Universal and nonuniversal properties of cross correlations in financial time series *Phys. Rev. Lett.* **83** 1471
[128] Moustakas A L, Baranger H U, Balents L, Sengupta A M and Simon S H 2000 Communication through a diffusive medium: coherence and capacity *Science* **287** 287 (*Preprint* cond-mat/0009097)